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Convexification and Concavification for a General Class of Global Optimization Problems

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Abstract. A kind of general convexification and concavification methods is proposed for solving some classes of global optimization problems with certain monotone properties. It is shown that these minimization problems can be transformed into equivalent concave minimization problem or reverse convex programming problem or canonical D.C. programming problem by using the proposed convexification and concavification schemes. The existing algorithms then can be used to find the global solutions of the transformed problems.

Key words: Concave minimization, D.C. programming, Global optimization, Monotone programming, Reverse convex programming

1. Introduction

We consider global optimization problems of the following form:

$$\begin{aligned} &\min f(x) \\ &\text{s.t. } g_i(x) \leqslant b_i, i = 1, \ldots, m \\ &x \in X, \end{aligned} \tag{1.1}$$

where $f : R^n \to R$ and $g_i : R^n \to R, i = 1, 2, \ldots, m$, are continuous functions satisfying certain monotone properties. Many practical global optimization problems possess monotone properties, such as the constraints reliability optimization problems, etc.

The problem (1.1) may have multiple local optimal solutions since $f(x)$ and g_is are not necessarily convex. Therefore, the standard optimization techniques fail by the existence of local minima that are not global. Due to the monotonicity of f and g_is, the optimal solution of (1.1) always lies on the boundary of the feasible region. Therefore, problem (1.1) is essentially a global optimization problem.

In recent years, a rapidly growing number of deterministic methods has been published for solving specific classes of multi-extremal global optimization problems, in particular, concave minimization, D.C. programming and reverse convex programming (see, e.g., [1–3, 9, 11]).

The main purpose of this paper is to present a kind of general convexification, concavification transformation methods to convert problem (1.1) into an equivalent structured problem, such as a concave minimization problem or a reverse convex minimization problem or a canonical D.C. programming problem. Then we can obtain the global solution of problem (1.1) by using the existing algorithms in [4] and [11]. Convexification solution schemes have been recently adopted successfully in some other subjects of optimization, such as in convexifying the perturbation function and Lagrangian function in the dual search methods for nonlinear programming (see, e.g., [5, 8]) and in convexifying the noninferior frontier in multiobjective optimization (see [6]). A special convexification (concavification) transformation for monotone function was developed in [7], and a class of convexification transformation for monotone function was proposed in [10]. However, the restrictive conditions under which the transformation can be successfully done limit the choice range of such transformation. This paper devotes to present a more general transformation including the above two as the special cases.

The paper is organized as follows. In Section 2, we state a basic theorem to transform the strictly monotone function into a convex(concave) one, and several corollaries. In Section 3, the convexification (concavification) transformation is applied to the functions in problem (1.1) in order to obtain an equivalent problem, such as concave minimization problem, reverse convex programming problem, or D.C. programming problem. By using the existing algorithms in [4] and [11], the successful search for a global optimal solution can be guaranteed. In Section 4, one illustrative example is presented to show how a problem with certain monotone properties can be transformed into an equivalent concave minimization problem.

2. Convexification (Concavification) of Monotone Functions

DEFINITION 2.1. We say that a function $h : R^n \to R$ is increasing (decreasing) on $D \subset R^n$ with respect to x_i *if*

$$h(x_1, \ldots, x_{i-1}, x_i^1, x_{i+1}, \ldots, x_n) \leqslant (\geqslant)\, h(x_1, \ldots, x_{i-1}, x_i^2, x_{i+1}, \ldots, x_n)$$

for $x_i^1 < x_i^2$; a function $h : R^n \to R$ is strictly increasing(decreasing) on $D \subset R^n$ with respect to x_i if

$$h(x_1, \ldots, x_{i-1}, x_i^1, x_{i+1}, \ldots, x_n) < (>)\, h(x_1, \ldots, x_{i-1}, x_i^2, x_{i+1}, \ldots, x_n)$$

for $x_i^1 < x_i^2$, where $x_i^1, x_i^2 \in D_i = \{x_i | (x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) \in D\}$.

DEFINITION 2.2. We say that a function $h(x) : R^n \to R$ is increasing (decreasing) if for any $x, y \in D$ with $x_i \leqslant y_i$ for $i = 1, \ldots, n$, it holds $h(x) \leqslant (\geqslant)\, h(y)$; a function $h(x)$ is strictly increasing (decreasing) if for any $x, y \in D$ with $x_i \leqslant y_i$ for $i = 1, \ldots, n$ and $x \neq y$ it holds $h(x) < (>) h(y)$.

DEFINITION 2.3. If the functions $f(x)$ and $g_i(x), i = 1, \ldots, m$, in problem (1.1) are all monotone (strictly monotone), then the problem (1.1) is called monotone programming (strictly monotone programming) problem.

DEFINITION 2.4. A real-valued function f defined on a convex set $X \subseteq R^n$ is called D.C. on X if, for all $x \in X, f$ can be expressed in the form

$$f(x) = p(x) - q(x)$$

where p and q are convex functions on X.

DEFINITION 2.5. A global optimization problem is called a concave minimization problem if it has the form (1.1), where X is a closed convex subset of R^n and the function f is a concave function and all functions, $g_i s$, are convex functions.

DEFINITION 2.6. A global optimization problem is called a reverse convex programming problem if it has the form (1.1), where X is a closed convex subset of R^n and the function f is a convex function and all functions, $g_i s$, are concave functions.

DEFINITION 2.7. A global optimization problem is called a D.C. programming problem or a D.C. program if it has the form (1.1), where X is a closed convex subset of R^n and all functions f and g_i are D.C. on X.

DEFINITION 2.8. A global optimization problem is called a canonical D.C. programming problem if it has the form (1.1), where X is a closed convex subset of R^n and the function f is a convex function, some of the functions, $g_i s$, are convex and the other $g_i s$ are concave.

Denote $(y_1^p, y_2^p, \ldots, y_n^p)$ by $y^p, (\ln(l + y_1^p), \ln(1 + y_2^p), \ldots, \ln(1 + y_n^p))$ by $\ln(1 + y^p)$, where $y_i > 0, i = 1, \ldots, m$.

Throughout the paper, we set

$$y \leqslant x \Leftrightarrow y_i \leqslant x_i, i = 1, \ldots, n,$$

$$y < x \Leftrightarrow y_i \leqslant x_i, i = 1, \ldots, n \text{ and } y \neq x.$$

Consider the following transformation of function $h(x)$:

$$h_p(y) = h(t_p(y)) \tag{2.1}$$

where $p > 0$ is a parameter, $t_p(y) : R^n \to R^n$ is a separable mapping, i.e., $t_p(y) = (t_{1,p}(y_1), t_{2,p}(y_2), \ldots, t_{n,p}(y_n))$ for $y = (y_1, \ldots, y_n)$. We further suppose that $t_{i,p}(y_i)$ is a 1-1 mapping. The domain of $h_p(y)$ is

$$Y_p = \left\{ y \in R^n | y_i = t_{i,p}^{-1}(x_i), (x^1, \ldots, x^n) \in X \right\} \tag{2.2}$$

Let Ω be an open set satisfying $Y_p \subseteq \Omega$ for all $p > 0$, $\Omega_i = \{y_i \in R | (y_1, \ldots, y_i, \ldots, y_n) \in \Omega\}$

THEOREM 2.1. *Suppose that*

(i) $h \in C^2(X), t_{i,p} \in C^2(\Omega_i), i = 1, \ldots, n;$

(ii) *there exists* $I \subset \{1, \ldots, n\}$ *such that* $h(x)$ *is strictly increasing on* X *with respect to any* $x_i, i \in I$ *and further satisfies*

$$\frac{\partial h(x)}{\partial x_i} \geqslant \eta_0, \quad \forall x \in X, \quad \forall i \in I, \tag{2.3}$$

where $\eta_0 > 0$ *is a constant;* $h(x)$ *is strictly decreasing on* X *with respect to any* $x_i, i \in \bar{I} = \{1, \ldots, n\} \backslash I$, *and further satisfies*

$$\frac{\partial h(x)}{\partial x_i} \leqslant -\zeta_0, \quad \forall x \in X, \quad \forall i \in \bar{I}, \tag{2.4}$$

where $\zeta_0 > 0$ *is a constant.*

(iii) $t_{i,p}, i = 1, \ldots, n$ *are strictly monotone functions on* Ω_i *satisfying*:

$$t'_{i,p}(y_i) \neq 0 \quad \forall y_i \in \Omega_i, \quad \forall i \in \{1, \ldots, n\} \tag{2.5}$$

$$\frac{t''_{i,p}(t^{-1}_{i,p}(x_i))}{[t'_{i,p}(t^{-1}_{i,p}(x_i))]^2} \to +\infty (p \to +\infty), \quad \forall x \in X, \quad \forall i \in I \tag{2.6}$$

$$\frac{t''_{i,p}(t^{-1}_{i,p}(x_i))}{[t'_{i,p}(t^{-1}_{i,p}(x_i))]^2} \to -\infty (p \to +\infty), \quad \forall x \in X, \quad \forall i \in \bar{I} \tag{2.7}$$

(iv) X *is a compact set.*

Then there exists a finite $p_0 > 0$ *such that* $h_p(y)$ *is a convex function on any convex subset of* Y_p *when* $p > p_0$.

Proof. Let $x = t_p(y), \forall \in Y_p$. By (2.1), we have

$$\frac{\partial h_p(y)}{\partial y_k} = \frac{\partial h(x)}{\partial x_k} t'_{k,p}(y_k),$$

$$\frac{\partial^2 h_p(y)}{\partial y_k^2} = \frac{\partial^2 h(x)}{\partial x_k^2} [t'_{k,p}(y_k)]^2 + \frac{\partial h(x)}{\partial x_k} t''_{k,p}(y_k)$$

$$= [t'_{k,p}(y_k)]^2 \left[\frac{\partial^2 h(x)}{\partial x_k^2} + \frac{\partial h(x)}{\partial x_k} \frac{t''_{k,p}(y_k)}{[t'_{k,p}(y_k)]^2} \right].$$

When $k \neq j$,

$$\frac{\partial^2 h_p(y)}{\partial y_k \partial y_j} = \frac{\partial^2 h(x)}{\partial x_k \partial x_j} t'_{k,p}(y_k) t'_{j,p}(y_j).$$

Let

$$A(x) = \text{diag}(t'_{1,p}(y_1), t'_{2,p}(y_2), \ldots, t'_{n,p}(y_n)), \tag{2.8}$$

$$B(x) = \mathrm{diag}\left(\frac{\partial h(x)}{\partial x_1}\frac{t''_{1,p}(y_1)}{[t'_{1,p}(y_1)]^2}, \frac{\partial h(x)}{\partial x_2}\frac{t''_{2,p}(y_2)}{[t'_{2,p}(y_2)]^2}, \ldots, \frac{\partial h(x)}{\partial x_n}\frac{t''_{n,p}(y_n)}{[t'_{n,p}(y_n)]^2}\right). \quad (2.9)$$

Denote the Hessian of $h(x)$ and $h_p(y)$ by $H(x)$ and $H_p(y)$, respectively. Then

$$H_p(y) = A(x)[H(x) + B(x)]A(x).$$

Let S^n be the unit sphere in R^n. For all $d \in S^n$,

$$d^{\mathrm{T}} H_p(y) d = d^{\mathrm{T}} A(x)[H(x) + B(x)]A(x)d. \quad (2.10)$$

Combining (2.5) and (2.8), we have that $A(x)[H(x) + B(x)]A(x)$ is a positive definite matrix if and only if $H(x) + B(x)$ is a positive definite matrix. For all $d \in S^n$,

$$d^{\mathrm{T}}[H(x) + B(x)]d = d^{\mathrm{T}} H(x) d + \sum_{i=1}^{n} \frac{\partial h(x)}{\partial x_i}\frac{t''_{i,p}(y_i)}{[t'_{i,p}(y_i)]^2} d_i^2.$$

Let $\tau_0 = \min \lambda_0(z)$, where $\lambda_0(z)$ denotes the minimum eigenvalue of $H(z)$. Suppose that $\tau_0 < 0$, otherwise $h(x)$ is convex already.

By (2.6) and (iii), (iv), we have that

$$\frac{t''_{i,p}(t^{-1}_{i,p}(x_i))}{[t'_{i,p}(t^{-1}_{i,p}(x_i))]^2} \rightrightarrows +\infty (p \to +\infty), \quad \forall x \in X, \quad \forall i \in I.$$

Thus, for $\frac{-\tau_0}{\eta_0} > 0$, there exists $p'_0 > 0$ such that for any $p > p'_0$,

$$\frac{t''_{i,p}(t^{-1}_{i,p}(x_i))}{[t'_{i,p}(t^{-1}_{i,p}(x_i))]^2} > \frac{-\tau_0}{\eta_0}, \quad \forall x \in X, \quad \forall i \in I.$$

By (2.7) and (iii), (iv), we have that

$$\frac{t''_{i,p}(t^{-1}_{i,p}(x_i))}{[t'_{i,p}(t^{-1}_{i,p}(x_i))]^2} \rightrightarrows -\infty (p \to +\infty), \quad \forall x \in X, \quad \forall i \in \bar{I}.$$

Thus, for $\frac{-\tau_0}{\zeta_0} > 0$, there exists $p''_0 > 0$ such that for any $p > p''_0$,

$$\frac{t''_{i,p}(t^{-1}_{i,p}(x_i))}{[t'_{i,p}(t^{-1}_{i,p}(x_i))]^2} < -\frac{-\tau_0}{\zeta_0}, \quad \forall x \in X, \quad \forall i \in \bar{I}.$$

Let $p_0 = \max\{p'_0, p''_0\}$, for any $p > p_0$, $d \in S^n$ and $x \in X$, we have

$$\begin{aligned} d^{\mathrm{T}}[H(x) + B(x)]d &\geqslant \tau_0 + \sum_{i=1}^{n} \frac{\partial h(x)}{\partial x_i}\frac{t''_{i,p}(y_i)}{[t'_{i,p}(y_i)]^2} d_i^2 \\ &> \tau_0 + \eta_0\left(-\frac{\tau_0}{\eta_0}\right)\sum_{i \in I} d_i^2 + (-\zeta_0)\frac{\tau_0}{\zeta_0}\sum_{i \in \bar{I}} d_i^2 \\ &= 0. \end{aligned}$$

Thus $H(x) + B(x)$ is a positive definite matrix, i.e. $A(x)[H(x) + B(x)]A(x)$ is a positive definite matrix when $p > p_0$. Therefore, $H_p(y)$ is a positive definite matrix for all $y \in Y_p$ when $p > p_0$. That is, $h_p(y)$ is a convex function on any convex subset of Y_p when $p > p_0$. □

REMARK 2.1. By the proof of Theorem 2.1, we know that $h_p(y)$ is convex if and only if $H(x) + B(x)$ is positive definite for any $x \in X$. Thus, if we can take proper p such that $H(x) + B(x)$ is positive definite, then $h_p(y)$ must be a convex function on Y_p. Furthermore, we need not to estimate the minimum value of the minimum eigenvalue of $H(z)$ for $z \in X$, we just need to take p large enough such that

$$\frac{\partial^2 h(x)}{\partial x_i^2} + \frac{\partial h(x)}{\partial x_i} \frac{t_i''(t_{i,p}^{-1}(x_i))}{[t_i'(t_{i,p}^{-1}(x_i))]^2} > \sum_{j=1, j\neq 1}^{n} \left| \frac{\partial^2 h(x)}{\partial x_i \partial x_i} \right|, \quad \text{for any } i = 1, \ldots, n,\ x \in X.$$

Then, the matrix $H(x) + B(x)$ must be a positive definite matrix on X.

COROLLARY 2.1. Let

(i) *$h(x)$ be a twice continuously differentiable and strictly increasing function on X satisfying*

$$\frac{\partial h(x)}{\partial x_i} \geqslant \eta_0, \quad \forall x \in X, \quad \forall i \in \{1, 2, \ldots, n\}; \tag{2.11}$$

(ii) *$t_{i,p}, i = 1, 2, \ldots, n$, be strictly monotone functions satisfying*

$$t_{i,p}'(y_i) \neq 0,\ \forall y_i \in \Omega_i, \quad \forall i \in \{1, \ldots, n\},$$

$$\frac{t_{i,p}''(t_{i,p}^{-1}(x_i))}{[t_{i,p}'(t_{i,p}^{-1}(x_i))]^2} \to +\infty (p \to +\infty), \quad \forall x \in X, \quad \forall i \in \{1, \ldots, n\}; \tag{2.12}$$

(iii) *X be a compact set.*

Then there exists a finite $p_1 > 0$, such that $h_p(y)$ is a convex function on any convex subset of Y_p when $p > p_1$.

COROLLARY 2.2. *Let*

(i) *$h(x)$ be a twice continuously differentiable and strictly decreasing function on X satisfying*

$$\frac{\partial h(x)}{\partial x_i} \leqslant -\zeta_0, \quad \forall x \in X, \quad \forall i \in \{1, 2, \ldots, n\}. \tag{2.13}$$

(ii) *$t_{i,p}, i = 1, 2, \ldots, n$, be strictly monotone functions satisfying*

$$t_{i,p}'(y_i) \neq 0, \quad \forall y_i \in \Omega_i, \quad \forall i \in \{1, \ldots, n\},$$

$$\frac{t_{i,p}''(t_{i,p}^{-1}(x_i))}{[t_{i,p}'(t_{i,p}^{-1}(x_i))]^2} \to -\infty (p \to +\infty), \quad \forall x \in X, \quad \forall i \in \{1, \ldots, n\}. \tag{2.14}$$

(iii) *X be a compact set.*

Then there exists a finite $p_2 > 0$, *such that* $h_p(y)$ *is a convex function on any convex subset of* Y_p *when* $p > p_2$.

COROLLARY 2.3. *Suppose that* h *and* $t_{i,p}, i = 1, \ldots, n$, *satisfy the conditions of Theorem (2.1),*

$$X = \{x | l_i \leqslant x_i \leqslant u_i, i = 1, \ldots, n\} \tag{2.15}$$

with $0 < l_i < u_i < \infty,\ i = 1, \ldots, n$. *Then there exists a finite* $p_3 > 0$ *such that* $h_p(y)$ *is a convex function on* Y_p *when* $p > p_3$.

Proof. We only need to verify that Y_p is a convex set. Let $J = \{j | t_{i,p}(y_j) \text{ is strictly increasing on } \Omega_j\}$, $\overline{J} = \{1, 2, \ldots, n\}/J$. Thus $Y_p = \{(y_1, \ldots, y_n) | t_{j,p}^{-1}(l_j) \leqslant y_j \leqslant t_{j,p}^{-1}(u_j), j \in J \text{ and } t_{j,p}^{-1}(u_j) \leqslant y_j \leqslant t_{j,p}^{-1}(l_j), j \in \overline{J}$. Obviously, Y_p is a convex compact set. □

COROLLARY 2.4. *Suppose* T_p *is a convex and strictly increasing function on a convex set* Z_0 *including* $h(X)$, h *and* t *satisfy the conditions of Theorem 2.1, let*

$$\Phi_p(y) = T_p(h(t_p(y))). \tag{2.16}$$

Then, $\Phi_p(y)$ *is a convex function on* Y_p.

Proof. By Theorem 2.1 and Theorem 6.9 ([12], pp. 154–155), we can obtain it very easily. □

REMARK 2.2. If we set $t_p(y) = \frac{1}{p} t(y)$, i.e. $t_{i,p}(y_i) = \frac{1}{p} t_i(y_i), i = 1, \ldots, n$, where $t_i(y_i)$ is a twice continuously differentiable and strictly monotone functions satisfying

$$\frac{t_i''(t_{i,p}^{-1}(x_i))}{[t_i'(t_{i,p}^{-1}(x_i))]^2} \geqslant \tau > 0, \forall x \in X, \forall i \in \{1, 2, \ldots, n\}, \tag{2.17}$$

and set $T_p(s) = T(ps), \forall s \in R$, where T is a strictly increasing convex function, then we obtain

$$\Phi_p(y) = T\left(ph(\frac{1}{p} t(y))\right) \tag{2.18}$$

which is exactly the transformation proposed in [10]. Obviously the transformation $t(y)$ satisfying the condition (2.17) must satisfy the condition (2.12) in Corollary 2.1. In fact, let $t_p(y) = \frac{1}{p} t(y)$, i.e. $\forall i \in \{1, 2, \ldots, n\}, t_{i,p}(y_i) = \frac{1}{p} t_i(y_i)$, then

$$t_{i,p}'(y_i) = \frac{1}{p} t_i'(y_i)$$

$$t_{i,p}''(y_i) = \frac{1}{p} t_i''(y_i).$$

If $t(y)$ satisfies the condition (2.17), then we have that

$$\frac{t''_{i,p}(t^{-1}_{i,p}(x_i))}{[t'_{i,p}(t^{-1}_{i,p}(x_i))]^2} = p\frac{t''_i(t^{-1}_{i,p}(x_i))}{[t'_i(t^{-1}_{i,p}(x_i))]^2}$$
$$\geqslant p\tau \to +\infty, (p \to +\infty), \forall x \in X, \forall i \in \{1, 2, \ldots, n\},$$

i.e $t_p(y)$ satisfies the condition (2.12) in Corollary 2.1.

Therefore the main result in [10] can be viewed as a special case of Corollary 2.1.

REMARK 2.3. If we set $t_p(y) = y^{\frac{1}{p}}$, i.e. $t_{i,p}(y_i) = y\, y_i^{\frac{1}{p}}$, $\forall i \in \{1, 2, \ldots, n\}$ and set $T_p(s) = s^p$ where $p > 0, s > 0, X$ be defined as in (2.15), moreover, without loss of generality we assume that $h(x) > 0, \forall x \in X$, then we obtain

$$\Phi_p(y) = [h(y^{\frac{1}{p}})]^p, \tag{2.19}$$

which is exactly the transformation proposed in [7]. We observe that condition (iii) in Corollary 2.2 is satisfied for this special class of transformation, thus the main result in [7] can be viewed as a special case of Corollary 2.2.

In fact, for each $i \in \{1, \ldots, n\}$, since $x_i = t_{i,p}(y_i) = y_i^{\frac{1}{p}}$, thus

$$Y_p = \{y \in R^n | 0 < l_i^p \leqslant y_i \leqslant u_i^p, i = 1, \ldots, n\},$$
$$t'_{i,p}(y_i) = \frac{1}{p} y_i^{\frac{1}{p}-1} \neq 0, \forall y_i > 0, \forall i \in \{1, 2, \ldots, n\}$$
$$t''_{i,p}(y_i) = \frac{1}{p}\left(\frac{1}{p} - 1\right) y_i^{\frac{1}{p}-2}, \forall i \in \{1, 2, \ldots, n\}$$

and

$$\frac{t''_{i,p}(t^{-1}_{i,p}(x_i))}{[t'_{i,p}(t^{-1}_{i,p}(x_i))]^2} = \frac{1-p}{x_i} (\text{when } p > 1)$$
$$\geqslant \frac{1-p}{l_i} \to -\infty (p \to +\infty), \forall x \in X, i \in \{1, \ldots, n\}.$$

REMARK 2.4. We can derive other transformations than those proposed in [7] and [10] from (2.1) by constructing many specific function forms which satisfy the conditions in Theorem 2.1. For example, if we take X as in (2.15), then each of functions $y_i^{-\frac{1}{p}}, \frac{1}{p}\ln(1 + y_i^{\frac{1}{p}}), \frac{1}{p}\ln(l + \frac{p}{y_i}), \ln(1 + y_i^{-\frac{1}{p}})$, $-\frac{1}{p}\ln(y_i)$ can be used as $t_{i,p}(y_i)$ satisfying

$$\frac{t''_{i,p}(t^{-1}_{i,p}(x_i))}{[t'_{i,p}(t^{-1}_{i,p}(x_i))]^2} \to +\infty (p \to +\infty), \forall x \in X,$$

and each of the functions $\frac{1}{p}\ln(1 + py_i), \frac{1}{p}\ln(1 + y_i^{\frac{1}{p}}), y_i^{\frac{1}{p}}, \ln(1 + y_i^{\frac{1}{p}}), \frac{1}{p}\ln(y_i)$ can be used as $t_{i,p}(y_i)$ satisfying

$$\frac{t''_{i,p}(t_{i,p}^{-1}(x_i))}{[t'_{i,p}(t_{i,p}^{-1}(x_i))]^2}$$

and each of the functions $s, s^p, \ln(1+e^{ps}), -\ln(1-\frac{1}{p}s)$ can be used as $T_p(s)$ satisfying condition in Corollary 2.4 provided p is sufficiently large (suppose $h(x) > 0$).

Similar to Theorem 2.1, we have the following concave transformation:

THEOREM 2.2. *Suppose the condition* (i) *and* (ii) *are the same as* (i) *and* (ii) *of Theorem 2.1.* (iii) $t_{i,p}, i = 1, \ldots, n$ *are strictly monotone functions on* Ω *satisfying*:

$$t'_{i,p}(y_i) \neq 0, \forall y_i \in \Omega_i, \quad \forall i \in \{1, \ldots, n\} \tag{2.20}$$

$$\frac{t''_{i,p}(t_{i,p}^{-1}(x_i))}{[t'_{i,p}(t_{i,p}^{-1}(x_i))]^2} \to -\infty (p \to +\infty), \quad \forall x \in X, \quad \forall i \in I \tag{2.21}$$

$$\frac{t''_{i,p}(t_{i,p}^{-1}(x_i))}{[t'_{i,p}(t_{i,p}^{-1}(x_i))]^2} \to +\infty (p \to +\infty), \quad \forall x \in X, \quad \forall i \in \bar{I} \tag{2.22}$$

(iv) X *is a compact set.*

Then there exists a finite $p_0 > 0$ *such that* $h_p(y)$ *is a concave function on any convex subset* of Y_p *when* $p > p_0$.

The proof of Theorem 2.2 is similar to the proof of Theorem 2.1, so it is omitted.

Similarly, Theorem 2.2 has the corresponding corollaries and remarks similar to those of Theorem 2.1. Here we do not enumerate them one by one.

By Theorem 2.1 and Theorem 2.2, we conclude that the function $h(x)$ which is strictly increasing or decreasing with respect to x_i for $i \in \{1, \ldots, n\}$ can always be transformed into a convex or concave function via the transformation (2.1). An important feature of the convexification (concavification)transformation (2.1) is that the variable transform $y \leftrightarrow t_p(y)$ is a 1–1 monotone mapping between Y_p and X which is crucial for the equivalence between problem (1.1) and the transformed one. These equivalences will be established in Section 3.

3. Equivalence to Concave Minimization, Reverse Convex Programming or D.C. Programming

In this section, we establish the equivalence between problem (1.1) and a transformed minimization problems with a better structure. Consider the following optimization problem which is a transformation of (1.1):

$$\min f_p(y) = f(t_p(y))$$
$$\text{s.t. } g_{j,p}(y) = g_j(t_p(y)) \leqslant b_j, j = 1, \ldots, m$$
$$y \in Y_p \tag{3.1}$$

where $t_p(y) : Y_P \to X$. The reference [10] pointed out the following conclusion of equivalence between (1.1) and (3.1).

THEOREM 3.1. [10] *Assume that $t_p(y)$ is an onto mapping with $X = \theta(Y_P)$.* Then:

(i) *y_p^* is a global optimal solution to* (3.1) *if and only if $x^* = t_p(y_p^*)$ is a global optimal solution to* (1.1).

(ii) *If t_p^{-1} exists and both t_p and t_p^{-1} are continuous mappings, then $y_p^* \in Y_p$ is a local optimal solution to* (3.1) *if and only if $x^* = t_p(y_p^*)$ is a local optimal solution to* (1.1).

Let $g_0(x) = f(x)$. In the remainder of this section, we suppose that X is a box defined by (2.15) and there exist two index sets $I \subset \{1, \ldots, n\}$ and $J \subset \{0, 1, \ldots, m\}$ and $\eta_0 > 0$ such that

for any $j \in J$

$$\frac{\partial g_j(x)}{\partial x_i} \geqslant \eta_0, \ \forall i \in I, \ x \in X \tag{3.2}$$

$$\frac{\partial g_j(x)}{\partial x_i} \leqslant -\eta_0, \ \forall i \in \bar{I} = \{1, \ldots, n\} \backslash I, x \in X \tag{3.3}$$

for any $j \in \bar{I}$

$$\frac{\partial g_j(x)}{\partial x_i} \leqslant -\eta_0 \ \forall i \in I, \ x \in X \tag{3.4}$$

$$\frac{\partial g_j(x)}{\partial x_i} \geqslant \eta_0, \ \forall i \in \bar{I} = \{1, \ldots, n\} \backslash I, \ x \in X. \tag{3.5}$$

And without loss of generality, we suppose that there exists at least a $g_j(x)$, $j \in \{1, \ldots, m\}$ such that the monotone properties of $g_j(x)$ and $f(x)$ are different, i.e., we must have that $J \neq \emptyset$ and $\bar{J} \neq \emptyset$. Thus it is enough to discuss the following three cases:

(1) $J = \{0\}$ and $\bar{J} = \{1, \ldots, m\}$. If we take $t_p(y)$ to satisfy the condition (iii) of Theorem 2.1, then, by Theorem 2.1, we know that the function $f_p(y)$ is convex on Y_p and by Theorem 2.2, functions $g_{j,p}(y)$, $j = 1, \ldots, m$, are concave on Y_p when p is large enough. Thus, the problem (3.1) is a reverse convex programming problem when p is large enough. If we take $t_p(y)$ to satisfy the condition (iii) of Theorem 2.2, then, by Theorem 2.2, we know that the function $f_p(y)$ is concave on Y_p and by Theorem 2.1, functions $g_{j,p}(y)$, $j = 1, \ldots, m$,

are convex on Y_p when p is large enough. Thus, the problem (3.1) is a concave minimization problem when p is sufficiently large.

(2) $J = \{1, \ldots, m\}$ and $\overline{J} = \{0\}$. If we take $t_p(y)$ to satisfy the condition (iii) of Theorem 2.1, then, by Theorem 2.1, we know that the functions $g_{j,p}(y)$, $j = 1, \ldots, m$, are convex on Y_p and by Theorem 2.2, function $f_p(y)$ is concave on Y_p when p is large enough. Thus, the problem (3.1) is a concave minimization problem when p is large enough. If we take $t_p(y)$ to satisfy the condition (iii) of Theorem 2.2, then, by Theorem 2.2, we know that the function $g_{j,p}(y)$, $j = 1, \ldots, m$, are concave on Y_p and by Theorem 2.1, function $f_p(y)$ is convex on Y_p when p is large enough. Thus, the problem (3.1) is a reverse convex programming problem when p is sufficiently large.

(3) $J \neq \{0\}$ and $\overline{J} \neq \{0\}$. Without loss of generality, we suppose that $0 \in J$. If we take $t_p(y)$ to satisfy the condition (iii) of Theorem 2.1, then, by Theorem 2.1, we know that the function $f_p(y)$ is convex on Y_p and by Theorem 2.2, functions $g_{j,p}(y)$, $j \in J$, are convex, $g_{j,p}(y)$, $j \in \overline{J}$, are concave on Y_p when p is large enough. Thus, the problem (3.1) is a canonical D.C. programming problem when p is large enough.

By using the existing algorithms for concave minimization, reverse convex programming and canonical D.C. programming (see [4, 11]), problem (3.1) can be solved, i.e. problem (1.1) can be solved successfully.

4. An Illustrative Example

In this section, one illustrative example is presented to show how a problem in the form of (1.1) can be transformed into an equivalent concave minimization problem.

EXAMPLE 4.1.

$$\begin{aligned} &\min \quad f(x) = 20x_1^4 - 30x_2^6 + \sin(5x_1)\sin(5x_2) \\ &\text{s.t. } g(x) = -10\exp(3x_1) + 10\exp(3x_2) - \cos(5x_1)\cos(5x_2) \leqslant 0, \\ &x \in X = \{x | 1 \leqslant x_1 \leqslant 2, 1 \leqslant x_2 \leqslant 2\}. \end{aligned} \tag{4.1}$$

Figures 1 and 2 give the behavior of function $f(x)$ and $g(x)$ on X. From Figures 1 and 2, we see that the function $f(x)$ and $g(x)$ are neither convex nor concave on X. In fact, we have that

$$\frac{\partial f(x)}{\partial x_1} = 80x_1^3 + 5\cos(5x_1)\sin(5x_2) \geqslant 80 - 5 = 75,$$
$$\frac{\partial f(x)}{\partial x_2} = -18x_2^5 + 5\sin(5x_1)\cos(5x_2) \leqslant -18 + 5 = -13,$$

and

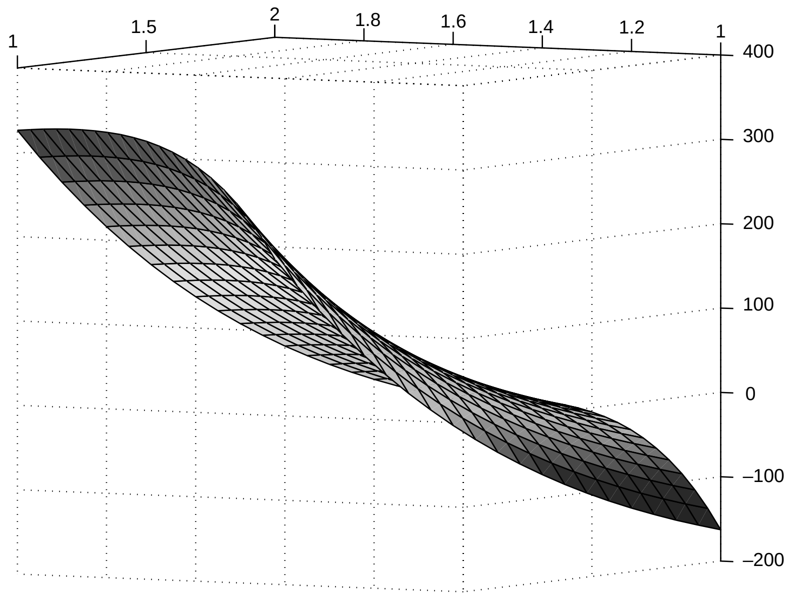

Figure 1. The behavior of function $f(x)$ on X.

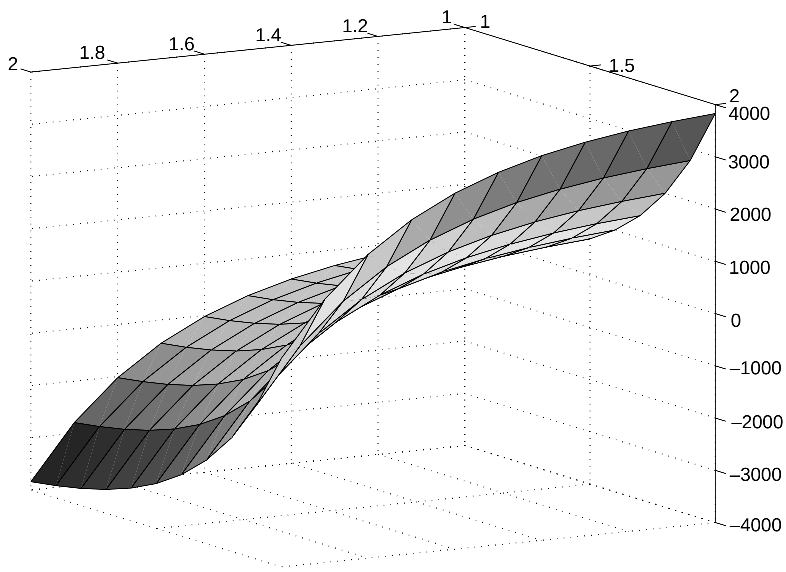

Figure 2. The behavior of function $g(x)$ on X.

$$\frac{\partial g(x)}{\partial x_1} = -30\exp(3x_1) + 5\sin(5x_1)\cos(5x_2) \leqslant -30\exp(3) + 5 < 0,$$
$$\frac{\partial g(x)}{\partial x_2} = 30\exp(3x_2) + 5\cos(5x_1)\sin(5x_2) \geqslant 30\exp(3) - 5 > 0,$$

for any $x \in X$.

And the Hessians of $f(x)$ and $g(x)$ at $x \in X$ are the following matrices, respectively.

$$H(x) = \begin{pmatrix} 240x_1^2 - 25\sin(5x_1)\sin(5x_2) & 25\cos(5x)\cos(5x_2) \\ 25\cos(5x)\cos(5x_2) & -90x_2^4 - 25\sin(5x_1)\sin(5x_2) \end{pmatrix}$$

and

$$G(x) = \begin{pmatrix} -90\exp(3x_1) + 25\cos(5x_1)\cos(5x_2) & -25\sin(5x_1)\sin(5x_2) \\ -25\sin(5x_1)\sin(5x_2) & 90\exp(3x_2) + 25\cos(5x_1)\cos(5x_2) \end{pmatrix}.$$

Obviously, they all are neither convex nor concave for any $x \in X$.

By Theorem 2.2 and 2.1, if can take $x = t_p(y)$ such that $H(x) + B_1(x)$ is negative definite and $G(x) + B_2(x)$ is positive definite, then the original problem (4.1) can be converted into an equivalent concave minimization problem, where

$$B_1(x) = \begin{pmatrix} \frac{\partial f(x)}{\partial x_1}\frac{t''_{1,p_1}(y_1)}{[t'_{1,p_1}(y_1)]^2} & 0 \\ 0 & \frac{\partial f(x)}{\partial x_2}\frac{t''_{2,p_2}(y_2)}{[t'_{2,p_2}(y_2)]^2} \end{pmatrix}$$

and

$$B_2(x) = \begin{pmatrix} \frac{\partial g(x)}{\partial x_1}\frac{t''_{1,p_1}(y_1)}{[t'_{1,p_1}(y_1)]^2} & 0 \\ 0 & \frac{\partial g(x)}{\partial x_2}\frac{t''_{2,p_2}(y_2)}{[t'_{2,p_2}(y_2)]^2} \end{pmatrix}.$$

Here if we take $x_1 = t_{1,p}(y_1) = \frac{1}{p}\ln(y_1)$ and take $x_2 = t_{2,p}(y_2) = y_2$, then the original problem (4.1) can be converted into the following problem:

$$\begin{aligned} &\min \quad f_p(y) = 20\left[\frac{1}{p}\ln(y_1)\right]^4 - 3y_2^6 + \sin\left(\frac{5}{p}\ln(y_1)\right)\sin(5y_2) \\ &\text{s.t. } g_p(y) = -10y_1^{\frac{3}{p}} + 10\exp(3y_2) - \cos\left(\frac{5}{p}\ln(y_1)\right)\cos(5y_2) \leqslant 0, \\ &y \in Y_p = \{e^p \leqslant y_1 \leqslant e^{2p}, 1 \leqslant y_2 \leqslant 2\}. \end{aligned} \tag{4.2}$$

We can easily verify that when $p \geqslant 4$, $H(z) + B_1(z)$ is negative definite and $G(z) + B_2(z)$ is positive definite for any $z \in Y_p$, i.e., the function $f_p(y)$ is a concave function and $g_p(y)$ is a convex function on Y_p (by Remark 2.1). Thus, the problem (4.2) is a concave minimization problem when $p \geqslant 4$.

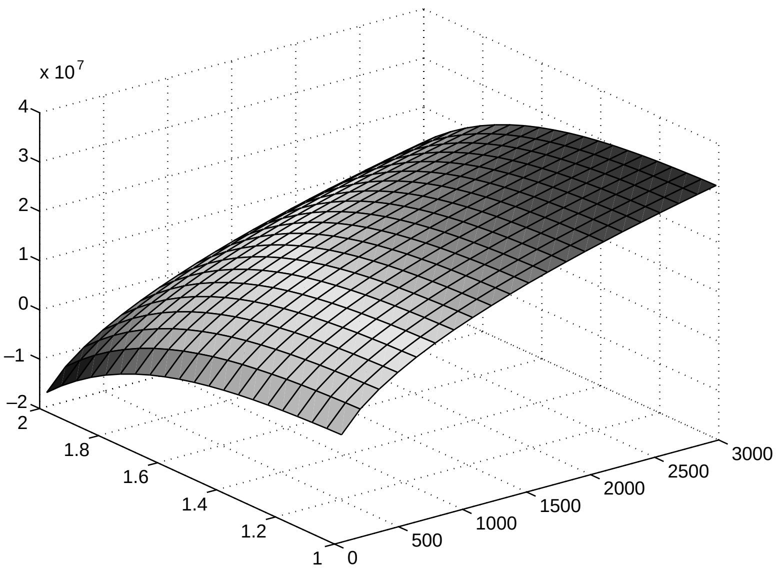

Figure 3. The behavior of function $f_p(y)$ $(p = 4)$ on X.

The figures 3 and 4 give the behavior of $f_p(y)$ and $g_p(y)$ on Y_p when $p = 4$.

From Figure 3 and 4, we see that the function $f_p(y)$ is concave and $g_p(y)$ is convex on Y_p. Therefore, the problem (4.1) has been converted into an equivalent concave minimization problem (4.2) by the above given trans-

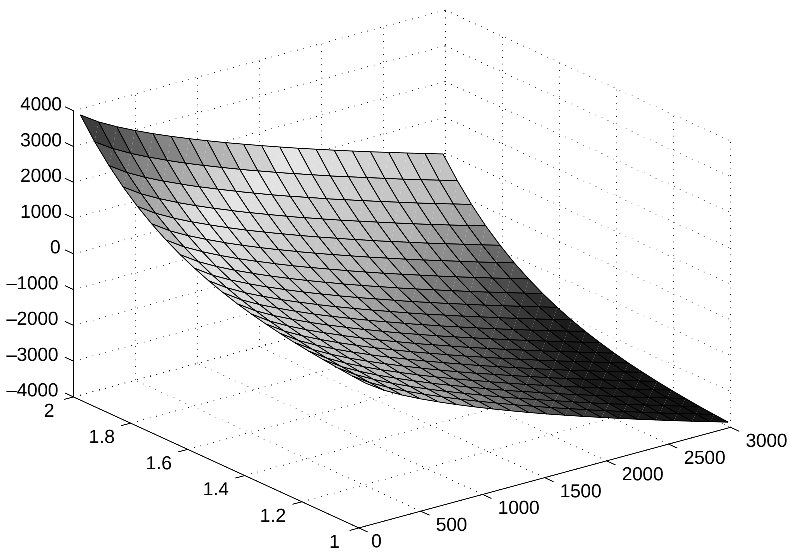

Figure 4. The behavior of function $g_p(y)$ $(p = 4)$ on X.

formation. Then by the existing algorithms for concave minimization, such as outer approximation method, etc., proposed in [2, 4, 11], we can obtain a global optimization solution of problem (4.1).

5. Conclusions

In this paper, we have given a general convexification and concavification transformation method to convert a general global optimization with certain monotone properties into an equivalent concave minimization, reverse convex programming problem or D.C. programming problem. The convexification and concavification transformation methods in [7] and [10] become special cases of the general convexification and concavification transformation proposed in this paper. Therefore, we can take more proper transformations to convert a general global optimization problem into an equivalent better structured problem. Then, we can obtain a global solution by solving the converted problem with the existing algorithms for these better structured problems.

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References

[1] Benson, H.P. (1996), Deterministic algorithm for constrained concave minimization: A unified critical surver. *Naval Research Logistics* 43, 765–795.
[2] Hoffman, K.L.A. (1981), A method for globally minimizing concave functions over convex set. *Mathematical Programing* 20, 22–23.
[3] Horst, R. (1990), Deterministic methods in constrained global optimization: Some recent advances and new fields of application. *Naval Research Logistics* 37, 433–471.
[4] Horst, R., Pardalos, P.M. and Thoai, N.V.(1996). Introduction to Global Optimization: Kluwer Academic Publishers, Dordrecht, Netherland.
[5] Li, D. (1995), Zero duality gap for a class of nonconvex optimization problems. *Journal of Optimization Theory and Applications* 85, 309–324.
[6] Li, D. (1996). Convexification of noninferior frontier. *Journal of Optimization Theory and Applications* 88, 177–196.
[7] Li, D. Sun, X.L. Biswal, M.P. and Gao, F. (2001) Convexification, concavification and monotonization in global optimization. *Annals of operations Research* 105, 213–226.
[8] Li, D. and Sun, X.L. (2000), Local convexification of Lagrangian function in nonconvex optimization. *Journal of Optimization Theory and Applications* 104, 109–120.
[9] Pardalos, P.M. and Rosen, J.B. (1987), *Constrained Global Optimization: Algorithms and Applications*, Springer-Verlag, Berlin.
[10] Sun, X.L., McKinnon, K. and Li, D. (2001) A convexification method for a class of global optimization problem with application to reliability optimization. *Journal of Global Optimization* 21, 185–199.

[11] Tuy, H. (1998). *Convex Analysis and Global Optimization*, Kluwer Academic Publishers, Dordrecht/Boston/London.
[12] Mordecai, A. (1976). *Nonlinear Programming: Analysis and Methods*, Prentice-hall International, Inc., London.